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Definable and invariant types in enrichments of NIP theories

Silvain Rideau and Pierre Simon*

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Let T be an NIP \mathcal{L} -theory and \tilde{T} be an enrichment. We give a sufficient condition on \tilde{T} for the underlying \mathcal{L} -type of every definable (respectively invariant) type over a model of \tilde{T} to be definable (respectively invariant) as an \mathcal{L} -type. Besides, we generalise work of Simon and Starchenko on the density of definable types among non forking types to this relative setting. These results are then applied to Scanlon's model completion of valued differential fields.

Let T be a theory in a language \mathcal{L} and consider an expansion $T \subseteq \tilde{T}$ in a language $\tilde{\mathcal{L}}$. In this paper, we wish to study how invariance and definability of types in T relate to invariance and definability of types in \tilde{T} . More precisely, let $\mathfrak{U} \models \tilde{T}$ be a monster model and consider some type $\tilde{p} \in S(\mathfrak{U})$ which is invariant over some small $M \models \tilde{T}$. Then the reduct p of \tilde{p} to \mathcal{L} is of course invariant under the action of the $\tilde{\mathcal{L}}$ -automorphisms of \mathfrak{U} that fix M (which we will denote as $\tilde{\mathcal{L}}(M)$ -invariant), but there is, in general, no reason for it to be $\mathcal{L}(M)$ -invariant. Similarly, if \tilde{p} is $\tilde{\mathcal{L}}(M)$ -definable, p might not be $\mathcal{L}(M)$ -definable.

When T is stable, and $\varphi(x; y)$ is an \mathcal{L} -formula, φ -types are definable by boolean combinations of instances of φ . It follows that if \tilde{p} is $\tilde{\mathcal{L}}(M)$ -invariant then p is both $\mathcal{L}(M)$ -invariant and $\mathcal{L}(M)$ -definable. Nevertheless, when T is only assumed to be NIP, then this is not always the case. For example one can take T to be the theory of dense linear orders and $\tilde{\mathcal{L}} = \{\leq, P(x)\}$ where $P(x)$ is a new unary predicate naming a convex non-definable subset of the universe. Then there is a definable type in \tilde{T} lying at some extremity of this convex set whose reduct to $\mathcal{L} = \{\leq\}$ is not definable without the predicate.

In the first section of this paper, we give a sufficient condition (in the case where T is NIP) to ensure that any $\tilde{\mathcal{L}}(M)$ -invariant (resp. definable) \mathcal{L} -type p is also $\mathcal{L}(M)$ -invariant (resp. definable). The condition is that there exists a model M of \tilde{T} whose reduct to \mathcal{L} is uniformly stably embedded in every elementary extension of itself. In the case where T is o-minimal for example, this happens whenever the ordering on M is complete.

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1 External separability

The main technical tool developed in this first section is the notion of external separability (Definition (1.2)). Two sets X and Y are said to be externally separable if there exists an externally definable set Z such that $X \subseteq Z$ and $Y \cap Z = \emptyset$. In Proposition (1.3), we show that in NIP theory, external separability is essentially a first order property. The results about definable and invariant sets then follow by standard methods along with a “local representation” of φ -types from [8].

In the second section, we generalise a result from [10] to this relative situation. We show that if T is dp-minimal and satisfies some technical assumption called property (D), then any \mathcal{L} -formula non-forking over M in the sense of \tilde{T} extends to an $\tilde{\mathcal{L}}(M)$ -definable \mathcal{L} -type. This result is somewhat orthogonal in strength to the results of Section 1: given a non forking, i.e. an invariant, $\tilde{\mathcal{L}}$ -type we build an arbitrarily close definable \mathcal{L} -type, but this type is *a priori* only $\tilde{\mathcal{L}}$ -definable unless we can also apply the results of Section 1.

The motivation for those results comes from the study of expansions of ACVF and in particular the model completion $\text{VDF}_{\mathcal{EC}}$ defined by Scanlon [6] of valued differential fields with a contractive derivation, i.e. a derivation ∂ such that for all x , $\text{val}(\partial(x)) \geq \text{val}(x)$. In the third section, we deduce, from the previous abstract results, a characterisation of definable (resp. invariant) types in models of $\text{VDF}_{\mathcal{EC}}$ in terms of the definability (resp. invariance) of the underlying ACVF-type. This characterisation also allows us to control the canonical basis of definable types in $\text{VDF}_{\mathcal{EC}}$, an essential step in proving elimination of imaginaries for that theory in [5].

0.1 Notations

Let us now define some notations that will be used throughout the paper. When $\varphi(x; y)$ is a formula, we implicitly consider that y is a parameter of the formula and we define $\varphi^{\text{opp}}(y; x)$ to be equal to $\varphi(x; y)$.

We write $M <^+ N$ to denote that M is a $|N|^+$ -saturated and (strongly) $|N|^+$ -homogenous elementary extension of N .

Let X be an $\mathcal{L}(M)$ -definable set (or a union of definable sets) and $A \subseteq M$. We denote by $X(A)$ the set of realisations of X in A , i.e. the set $\{a \in A : M \models a \in X\}$. If \mathcal{R} is a set of definable sets (in particular a set of sorts), we define $\mathcal{R}(A) := \bigcup_{R \in \mathcal{R}} R(A)$.

Finally if p and q are (global) invariant types, we denote by $p \otimes q$ the unique type such that for all set C , $(a, b) \models p \otimes q|_C$ if and only if $b \models q|_C$ and $a \models p|_{Cb}$.

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Definition 1.1:

<Externally φ -definable> Let M be an \mathcal{L} -structure, $\varphi(x; t)$ be an \mathcal{L} -formula and X a subset of some cartesian power of M . We say that X is externally φ -definable if there exist $N \geq M$ and a tuple $a \in N$ such that $X = \varphi(M; a)$.

Definition 1.2:

<Externally φ -separable> Let M be an \mathcal{L} -structure, $\varphi(x; t)$ be an \mathcal{L} -formula and X, Y be

1 External separability

subsets of some cartesian power of M . We say that X and Y are externally φ -separable if there exist $N \geq M$ and a tuple $a \in N$ such that $X \subseteq \varphi(M; a)$ and $Y \cap \varphi(M; a) = \emptyset$.

We will say that X and Y are φ -separable if a can be chosen in M . Note that a set X is externally φ -definable if X and its complement are externally φ -separable.

Proposition 1.3:

Let T be an \mathcal{L} -theory and $\varphi(x; t)$ an NIP \mathcal{L} -formula. Let $U(x)$ and $V(x)$ be new predicate symbols and let $\mathcal{L}_{U,V} := \mathcal{L} \cup \{U, V\}$. Then, there is an $\mathcal{L}_{U,V}$ -sentence $\theta_{U,V}$ and an \mathcal{L} -formula $\psi(x; s)$ such that for all $M \models T$ and any enrichment $M_{U,V}$ of M to $\mathcal{L}_{U,V}$, we have:

if U and V are externally φ -separable, then $M_{U,V} \models \theta_{U,V}$

and

if $M_{U,V} \models \theta_{U,V}$, then U and V are externally ψ -separable.

Proof. Let k_1 be the VC-dimension of $\varphi(x; t)$. By the dual version of the (p, q) -theorem (see [7, Corollary 6.13]) there exists q_1 and n_1 such that for any set X , any finite $A \subseteq X$ and any $\mathcal{S} \subseteq \mathfrak{P}(X)$ of VC-dimension at most k_1 , if for all $A_0 \subseteq A$ of size at most q_1 there exists $S \in \mathcal{S}$ containing A_0 , then there exists $S_1 \dots S_{n_1} \in \mathcal{S}$ such that $A \subseteq \bigcup_{i \leq n_1} S_i$. Let k_2 be the VC-dimension of $\bigcup_{i=1}^{n_1} \varphi(x; t_i)$ and q_2 and n_2 the bounds obtained by the dual (p, q) -theorem for families of VC-dimension at most k_2 . Let

$$\theta_{U,V} := \forall x_1 \dots x_{q_1}, y_1 \dots y_{q_2} \bigwedge_{i \leq q_1} U(x_i) \wedge \bigwedge_{j \leq q_2} V(y_j) \Rightarrow \exists s \bigwedge_{i \leq q_1} \varphi(x_i; t) \wedge \bigwedge_{i \leq q_2} \neg \varphi(y_i; t).$$

Now, let $M \leq N \models T$, U and V be subsets of $M^{|x|}$ and $d \in N$ be a tuple. If $U \subseteq \varphi(M; d)$ and $V \subseteq \neg \varphi(M; d)$ then for any $A \subseteq U$ and $B \subseteq V$ finite there exists $d_0 \in M$ such that $A \subseteq \varphi(M; d_0)$ and $B \subseteq \neg \varphi(M; d_0)$. In particular, $M_{U,V} \models \theta_{U,V}$.

Suppose now that $M_{U,V} \models \theta_{U,V}$. Let $B_0 \subseteq V$ have cardinality at most q_2 . The family $\{\varphi(M; d) : d \in M \text{ a tuple and } B_0 \subseteq \neg \varphi(M; d)\}$ has VC-dimension at most k_1 and — because $M_{U,V} \models \theta_{U,V}$ — for any $A_0 \subseteq U$ of size at most q_1 , there exists $d \in M$ such that $A_0 \subseteq \varphi(M; d)$ and $B_0 \subseteq \neg \varphi(M; d)$. It follows that for any finite $A \subseteq U$ there are tuples $d_1 \dots d_{n_1} \in M$ such that $A \subseteq \bigvee_{i \leq n_1} \varphi(M; d_i)$ and for all $i \leq n_1$, $B_0 \subseteq \neg \varphi(M; d_i)$, in particular, $B_0 \subseteq \neg (\bigvee_{i \leq n_1} \varphi(M; d_i))$. By compactness, there exists tuple $d_1 \dots d_{n_1} \in N \geq M$ such that $U \subseteq \bigvee_{i \leq n_1} \varphi(M; d_i)$ and $B_0 \subseteq \neg (\bigvee_{i \leq n_1} \varphi(M; d_i))$.

The family $\{\neg (\bigvee_{i \leq n_1} \varphi(M; d_i)) : d_i \in N \leq M \text{ tuples and } U \subseteq \bigvee_{i \leq n_1} \varphi(M; d_i)\}$ has VC-dimension at most k_2 . We have just shown that for any B_0 of size at most q_2 , there is an element of that family containing B_0 . It follows by the (p, q) -property and compactness that there exists tuples $d_{i,j} \in N \geq M$ such that $V \subseteq \bigvee_{j \leq n_2} \neg (\bigvee_{i \leq n_1} \varphi(M; d_{i,j})) = \neg (\bigwedge_{j \leq n_2} \bigvee_{i \leq n_1} \varphi(M; d_{i,j}))$ and $U \subseteq \bigwedge_{j \leq n_2} \bigvee_{i \leq n_1} \varphi(M; d_{i,j})$. Hence U and V are externally $\bigwedge_{j \leq n_2} \bigvee_{i \leq n_1} \varphi(x; t_{i,j})$ -separable. \blacksquare

We would now like to characterise enrichments \tilde{T} of NIP theories that do not add new externally separable definable sets, i.e. $\tilde{\mathcal{L}}$ -definable sets that are externally \mathcal{L} -separable but not internally \mathcal{L} -separable. We show that if there is one model of \tilde{T} where this property holds uniformly, then it holds in all models of T .

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Proposition 1.4:

Let T be an NIP \mathcal{L} -theory (with at least two constants), $\tilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language, $\tilde{T} \supseteq T$ be a complete theory and $\chi_1(x; s)$ and $\chi_2(x; s)$ be $\tilde{\mathcal{L}}$ -formulas. The following are equivalent:

- (i) For all \mathcal{L} -formula $\varphi(x; t)$, all $M \models \tilde{T}$ and all $a \in M$ there exists an \mathcal{L} -formula $\xi(x; z)$ such that if $\chi_1(M; a)$ and $\chi_2(M; a)$ are externally φ -separated then they are ξ -separated;
- (ii) For all \mathcal{L} -formula $\varphi(x; t)$, there exists an \mathcal{L} -formula $\xi(x; z)$ such that for all $M \models \tilde{T}$ and all $a \in M$, if $\chi_1(M; a)$ and $\chi_2(M; a)$ are externally φ -separated then they are ξ -separated;
- (iii) For all \mathcal{L} -formula $\varphi(x; t)$, there exists an \mathcal{L} -formula $\xi(x; z)$ and $M \models \tilde{T}$ such that for all $a \in M$, if $\chi_1(M; a)$ and $\chi_2(M; a)$ are externally φ -separated then they are ξ -separated.

Proof. The implications (ii) \Rightarrow (i) and (ii) \Rightarrow (iii) are trivial.

Let us now show that (iii) implies (ii). By Proposition (1.3), there exists an $\tilde{\mathcal{L}}$ -formula $\theta(s)$ and an \mathcal{L} -formula $\psi(x; u)$ such that for all $N \models \tilde{T}$ and $a \in N$:

$$\chi_1(N; a) \text{ and } \chi_2(N; a) \text{ externally } \varphi\text{-separated implies } N \models \theta(a)$$

and

$$N \models \theta(a) \text{ implies } \chi_1(N; a) \text{ and } \chi_2(N; a) \text{ externally } \psi\text{-separated.}$$

Let M and ξ be as in condition (iii) with respect to ψ . We have:

$$M \models \forall s \theta(s) \Rightarrow \exists u (\forall x (\chi_1(x; s) \Rightarrow \xi(x; u)) \wedge (\chi_2(x; s) \Rightarrow \neg \xi(x; u))).$$

As \tilde{T} is complete, this must hold in any $N \models \tilde{T}$. Thus, if $\chi_1(N; a)$ and $\chi_2(N; a)$ are externally φ -separated, we have $N \models \theta(a)$ and hence $\chi_1(N; a)$ and $\chi_2(N; a)$ are ξ -separated. There remains to prove that (i) \Rightarrow (iii). Pick any $M \models \tilde{T}$ and let $\mathfrak{U} \succ M$ be $|M|^+$ -saturated. By (i), it is impossible to find, in any elementary extension (\mathfrak{U}^*, M^*) of the pair (\mathfrak{U}, M) , a tuple $a \in M^*$ and $b \in \mathfrak{U}^*$ such that $\chi_1(M^*; a)$ and $\chi_2(M^*; a)$ are separated by $\varphi(M^*; b)$, but they are not separated by any set of the form $\xi(M^*; c)$ where ξ is an \mathcal{L} -formula and $c \in M^*$. By compactness, there exists $\xi_i(x; u_i)$ for $i \leq n$ such that for all $a \in M$ if $\chi_1(M; a)$ and $\chi_2(M; a)$ are externally φ -separated, there exists an i such they are ξ_i -separated. By classic coding tricks, we can ensure that $i = 1$. \blacksquare

Definition 1.5:

<Uniform stable embeddedness> Let M be an \mathcal{L} -structure and $A \subseteq M$. We say that A is uniformly stably embedded in M if for all formulas $\varphi(x; t)$ there exists a formula $\chi(x; s)$ such that for all tuples $b \in M$ there exists a tuple $a \in A$ such that $\varphi(A, b) = \chi(A, a)$.

Remark 1.6:

If there exists $M \models \tilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension, then such an M witnesses Condition 1.4.(iii) for every choice of formulas χ_1 and χ_2 .

Corollary 1.7:

Let T be an NIP \mathcal{L} -theory that eliminates imaginaries, $\tilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language and $\tilde{T} \supseteq T$ be a complete $\tilde{\mathcal{L}}$ -theory. Suppose that there exists $M \models \tilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension. Let $\varphi(x; t)$ be an \mathcal{L} -formula, $N \models \tilde{T}$, $A = \text{dcl}_{\tilde{\mathcal{L}}}^{\text{eq}}(A) \subseteq N^{\text{eq}}$ and $p \in \mathcal{S}_x^{\varphi}(N)$. If p is $\tilde{\mathcal{L}}^{\text{eq}}(A)$ -definable, then it is in fact $\mathcal{L}(\mathcal{R}(A))$ -definable where \mathcal{R} denotes the set of all \mathcal{L} -sorts.

Proof. Let $a \models p$. Then $X := \{m \in N : \varphi(x; m) \in p\} = \{m \in N : \models \varphi(a; m)\}$ is \mathcal{L} -externally definable and $\tilde{\mathcal{L}}^{\text{eq}}(A)$ -definable (by some $\tilde{\mathcal{L}}$ -formula χ). It follows from Remark (1.6) that Condition 1.4.(iii) holds and hence, by Condition 1.4.(i), taking $\chi_1 = \chi$ and $\chi_2 = \neg\chi$, it follows that X is \mathcal{L} -definable.

As for X being $\mathcal{L}(\mathcal{R}(A))$ -definable, because T eliminates imaginaries, we have just shown that we can find $\ulcorner X \urcorner^{\mathcal{L}} \in \mathcal{R}$. But X is also $\tilde{\mathcal{L}}^{\text{eq}}(A)$ -definable and hence $\ulcorner X \urcorner^{\mathcal{L}} \in \text{dcl}_{\tilde{\mathcal{L}}}^{\text{eq}}(A) = A$, i.e. $\ulcorner X \urcorner^{\mathcal{L}} \in A \cap \mathcal{R} = \mathcal{R}(A)$. \blacksquare

We will need the following fact, which is [8, Proposition 2.11].

Fact 1.8:

Let T be any theory, $\varphi(x; y)$ an NIP formula, $M <^+ N \models T$ and $p(x)$ a global M -invariant φ -type. Let $b, b' \in \mathfrak{U} \preceq N$ such that both $\text{tp}(b/N)$ and $\text{tp}(b'/N)$ are finitely satisfiable in M and $\text{tp}_{\varphi^{\text{opp}}}(b/N) = \text{tp}_{\varphi^{\text{opp}}}(b'/N)$. Then we have $p|_{\mathfrak{U}} \vdash \varphi(x; b) \iff \varphi(x; b')$.

Proposition 1.9:

Let T be an NIP \mathcal{L} -theory, $\tilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language and $\tilde{T} \supseteq T$ be a complete $\tilde{\mathcal{L}}$ -theory. Let \mathcal{R} denote the set of \mathcal{L} -sorts. Suppose that there exists $M \models \tilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension. Let $\varphi(x; t)$ be an \mathcal{L} -formula, $N \models \tilde{T}$ be sufficiently saturated, $A \subseteq N$ and $p \in \mathcal{S}_x^{\varphi}(N)$ be $\tilde{\mathcal{L}}(A)$ -invariant. Assume that every $\tilde{\mathcal{L}}(A)$ -definable sets (in some cartesian power of \mathcal{R}) is consistent with some global $\mathcal{L}(\mathcal{R}(A))$ -invariant type. Then p is $\mathcal{L}(A)$ -invariant.

Proof. Let us first assume that $A \models \tilde{T}$. Let b_1 and b_2 be such $p(x) \vdash \varphi(x, b_1) \wedge \neg\varphi(x, b_2)$. We have to show that $\text{tp}_{\mathcal{L}}(b_1/A) \neq \text{tp}_{\mathcal{L}}(b_2/A)$. Let $p_i = \text{tp}_{\tilde{\mathcal{L}}}(b_i/A)$, $\Sigma(t)$ be the set of $\tilde{\mathcal{L}}(N)$ -formulas $\theta(t)$ such that $\neg\theta(A) = \emptyset$ and $\Delta(t_1, t_2)$ be the set:

$$p_1(t_1) \cup p_2(t_2) \cup \Sigma(t_1) \cup \Sigma(t_2) \cup \{\varphi(n, t_1) \iff \varphi(n, t_2) : n \in N\}.$$

If Δ were consistent, there would exist b_1^* and b_2^* such that $b_i \equiv_{\tilde{\mathcal{L}}(A)} b_i^*$, $\text{tp}(b_i^*/N)$ finitely satisfiable in A and $\text{tp}_{\varphi^{\text{opp}}}(b_1^*/N) = \text{tp}_{\varphi^{\text{opp}}}(b_2^*/N)$. Applying Fact (1.8) it would follow that $p(x) \vdash \varphi(x; b_1^*) \iff \varphi(x; b_2^*)$. But, because p is $\tilde{\mathcal{L}}(A)$ -invariant and $p(x) \vdash \varphi(x, b_1) \wedge \neg\varphi(x, b_2)$, we also have that $p(x) \vdash \varphi(x; b_1^*) \wedge \neg\varphi(x; b_2^*)$, a contradiction.

By compactness, there exists $\psi_i \in p_i$, $\theta_i \in \Sigma$, $n \in \omega$ and $(c_i)_{i \in n}$ such that

$$\forall t_1, t_2 \theta_1(t_1) \wedge \theta_2(t_2) \wedge (\bigwedge_i \varphi(c_i, t_1) \iff \varphi(c_i, t_2)) \wedge \psi_1(t_1) \Rightarrow \neg\psi_2(t_2).$$

In particular, because $\neg\theta_i(A) = \emptyset$, for all m_1 and $m_2 \in A$, $(\bigwedge_i \varphi(c_i, m_1) \iff \varphi(c_i, m_2)) \wedge \psi_1(m_1) \Rightarrow \neg\psi_2(m_2)$. For all $\varepsilon : n \rightarrow 2$, let $\varphi_{\varepsilon}(t, c) := \bigwedge_i \varphi(c_i, t)^{\varepsilon(i)}$ where $\varphi^1 = \varphi$ and

2 Non-forking types and definable types

$\varphi^0 = \neg\varphi$. It follows that if $\varphi_\varepsilon(A, c) \cap \psi_1(A) \neq \emptyset$, then $\varphi_\varepsilon(A, c) \cap \psi_2(A) = \emptyset$. Let

$$\theta(t, c) := \bigwedge_{\varphi_\varepsilon(A, c) \cap \psi_1(A) \neq \emptyset} \varphi_\varepsilon(c, t).$$

We have $\psi_1(A) \subseteq \theta(A, c)$ and $\psi_2(A) \cap \theta(A, c) = \emptyset$, i.e. $\psi_1(A)$ and $\psi_2(A)$ are externally θ -separable. By Proposition (1.4) and Remark (1.6), $\psi_1(A)$ and $\psi_2(A)$ are in fact ξ -separable for some $\mathcal{L}(A)$ -formula ξ . It follows that $N \models \forall t_1, t_2 (\psi_1(t_1) \Rightarrow \xi(t_1)) \wedge (\psi_2(t_2) \Rightarrow \neg\xi(t_2))$ and, in particular $N \models \xi(b_1) \wedge \neg\xi(b_2)$.

Let us now conclude the proof when A is not a model. Let $M \models \tilde{T}$ contain A and pick any a and $b \in N$ such that $a \equiv_{\mathcal{L}(\mathcal{R}(A))} b$.

Claim 1.10: *There exists $M^* \equiv_{\tilde{\mathcal{L}}(A)} M$ (in particular it is a model of \tilde{T} containing A) such that $a \equiv_{\mathcal{L}(\mathcal{R}(M^*))} b$.*

Proof. By compactness, it suffices given $\chi(y) \in \text{tp}_{\tilde{\mathcal{L}}}(M/A)$ and $\psi_i(t; y)$ a finite number of \mathcal{L} -formulas, to find a tuple m such that $\models \chi(m) \wedge \bigwedge_i \psi(a; m) \iff \psi(b; m)$. Projecting χ onto its \mathcal{R} -variables, we may assume that y is a tuple of \mathcal{R} -variables. By hypothesis on A , there exists $q \in \mathcal{S}_y(N|_{\mathcal{L}})$ which is $\mathcal{L}(\mathcal{R}(A))$ -invariant and consistent with χ . Then any $m \models q|_{\mathcal{R}(A)ab} \cup \{\chi(y)\}$ has the required properties. \blacklozenge

As p is $\tilde{\mathcal{L}}(A)$ -invariant it is in particular $\tilde{\mathcal{L}}(M^*)$ -invariant. But, as shown above, p is then $\mathcal{L}(M^*)$ -invariant. It follows that $p \vdash \varphi(x; a) \iff \varphi(x; b)$. \blacksquare

The assumption that all $\tilde{\mathcal{L}}(A)$ -definable sets are consistent with some global $\mathcal{L}(A)$ -invariant type may seem like a surprising assumption. Nevertheless, considering a coheir (in the sense of \tilde{T} , whose restriction to \mathcal{L} is also a coheir in the sense of T), this assumption always holds when A is a model of \tilde{T} .

2 Non-forking types and definable types

The goal of this section is to show that results of [9, 10] concerning definable types in dp-minimal theories extend *mutatis mutandis* to our relative setting.

We start by recalling some notions about NIP theories exactly as in [10]. Let M be a model of an NIP theory. A sequence $(b_i)_{i < \omega}$ is strictly non-forking over M if for each $i < \omega$, $\text{tp}(b_i/b_{<i}M)$ is strictly non-forking over M which means that it extends to a global type $\text{tp}(b_*/\mathfrak{U})$ such that both $\text{tp}(b_*/\mathfrak{U})$ and $\text{tp}(\mathfrak{U}/Mb_*)$ are non-forking over M . We will only need to know two facts about strict non-forking sequences (both proved in [1], see also [7, Chapter 5]):

(Existence) Given $b \in \mathfrak{U}$ and $M \models T$, there is an indiscernible sequence $b = b_0, b_1, \dots$ which is strictly non-forking over M . We call such a sequence a *strict Morley sequence* of $\text{tp}(b/M)$.

(Witnessing property) If the formula $\varphi(x; b)$ forks over M , then for any strictly non-forking indiscernible sequence $b = b_0, b_1, \dots$, the type $\{\varphi(x; b_i) : i < \omega\}$ is inconsistent.

2 Non-forking types and definable types

If $\varphi(x; y)$ is an NIP formula, we let $\text{alt}(\varphi)$ be the *alternation number* of φ , namely the maximal n for which there is an indiscernible sequence $(b_i : i < \omega)$ and a tuple a with $\neg(\varphi(a; b_i) \leftrightarrow \varphi(a; b_{i+1}))$ for all $i < n$. If $(b_i : i < \omega)$ is indiscernible and $\{\varphi(x; b_i) : i < \text{alt}(\varphi)/2 + 1\}$ is consistent, then $\{\varphi(x; b_i) : i < \omega\}$ is also consistent.

We will also need the notion of “ b -forking” as defined in Cotter and Starchenko’s paper [2] and as recalled in [9]. For this, we assume that T is NIP.

Assume we have $M \prec^+ N$ and $b \in \mathfrak{U}$ such that $\text{tp}(b/N)$ is M -invariant. We say that a formula $\psi(x, b; d) \in L(Nb)$ b -divides over M if there is an M -indiscernible sequence $(d_i : i < \omega)$ inside N with $d_0 = d$ and $\{\psi(x, b; d_i) : i < \omega\}$ is inconsistent. We define b -forking in the natural way.

Fact 2.1 (T is NIP):

Notations being as above, the following are equivalent:

- (i) $\psi(x, b; d)$ does not b -divide over M ;
- (ii) $\psi(x, b; d)$ does not b -fork over M ;
- (iii) if $(d_i : i < \omega)$ is a strict Morley sequence of $\text{tp}(d/M)$ inside N , then $\{\psi(x, b; d_i) : i < \omega\}$ is consistent;
- (iii)’ if $(d_i : i < \omega)$ is a strict Morley sequence of $\text{tp}(d/M)$ inside N , then $\{\psi(x, b; d_i) : i < m\}$ is consistent where m is greater than the alternation number of $\psi(x, y; z)$;
- (iv) there is $a \models \psi(x, b; d)$ such that $\text{tp}(a, b/N)$ is M -invariant.

We will say that T has property (D) if for every set A and consistent formula $\varphi(x) \in L(A)$, with x a single variable, there is an A -definable complete type $p \in S_x(A)$ extending $\varphi(x)$. Note that we do not assume that p extends to a global A -definable type. Property (D) holds in ACVF (cf. [10, Proposition 7]).

In what follows, we consider a complete theory T in a language \mathcal{L} along with some \tilde{T} extending T in a language $\tilde{\mathcal{L}} \supseteq \mathcal{L}$. We are mainly interested in the case where \tilde{T} is NIP and T is dp-minimal with property (D), but we will assume this only when necessary.

Lemma 2.2:

Let $M \prec N$ be models of \tilde{T} and $b \in \mathfrak{U}$ such that $\text{tp}_{\mathcal{L}}(b/N)$ is $\tilde{\mathcal{L}}(M)$ -definable. Assume that $p \in S_x^{\mathcal{L}}(Mb)$ is an $\tilde{\mathcal{L}}(Mb)$ -definable \mathcal{L} -type, then p extends to some $q \in S_x^{\mathcal{L}}(Nb)$ which is $\tilde{\mathcal{L}}(Mb)$ -definable using the same definition scheme as p .

Proof. For each formula $\varphi(x; y, b) \in \mathcal{L}(b)$, there is, by hypothesis, a formula $d\varphi(y; b) \in \tilde{\mathcal{L}}(M)$ such that for every $d \in M^{|y|}$ we have $p \vdash \varphi(x; d, b)$ if and only if $\mathfrak{U} \models d\varphi(d; b)$. We have to check that the scheme $\varphi(x; y, b) \mapsto d\varphi(y; b)$ defines a consistent complete type over Nb . Let us check completeness for example. Assume that there is some $n \in N$ and formula $\varphi(x; y, b)$ such that $\mathfrak{U} \models \neg d\varphi(n; b) \wedge \neg d(\neg\varphi)(n; b)$. Then by definability of $\text{tp}_{\mathcal{L}}(b/N)$ over M , we can find such a n in M ; contradiction. Consistency is proved in the same way. ■

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Lemma 2.3:

<T is NIP> Let $M \prec^+ N$, $n < \omega$ and assume that for any formula $\theta(y; d) \in \tilde{\mathcal{L}}(N)$ with $|y| = n$ and non-forking over M , there is an \mathcal{L} -type over N , which is $\tilde{\mathcal{L}}(M)$ -definable and consistent with $\theta(y; d)$. Let $\varphi(x, y; d) \in \tilde{\mathcal{L}}(N)$ be non-forking over M , where $|y| = n$ and $|x| = 1$. Then we can find a tuple $(a, b) \models \varphi(x, y; d)$ such that $\text{tp}_{\tilde{\mathcal{L}}}(a, b/N)$ is $\tilde{\mathcal{L}}(M)$ -invariant and $\text{tp}_{\mathcal{L}}(b/N)$ is $\tilde{\mathcal{L}}(M)$ -definable.

Proof. Let $(d_i : i < \omega)$ be a strict Morley sequence of $\text{tp}(d/M)$ inside N . Let $m < \omega$ be greater than the alternation number of $\varphi(x, y; z)$. As the formula $\varphi(x, y; d)$ does not fork over M , it extends to a global $\tilde{\mathcal{L}}(M)$ -invariant type p . Then the conjunction $\psi(x, y; \bar{d}) = \bigwedge_{i < m} \varphi(x, y; d_i)$ is in p . In particular it is consistent and does not fork over M . The same is true for $\theta(y; \bar{d}) = (\exists x)\psi(x, y; \bar{d})$. By hypothesis, we can find some $b \in \mathfrak{U}$ such that $\text{tp}_{\mathcal{L}}(b/N)$ is $\tilde{\mathcal{L}}(M)$ -definable and $\mathfrak{U} \models \theta(b; \bar{d})$. We claim that the formula $\varphi(x; b, \bar{d})$ does not b -fork over M . Assume that it did. Then the conjunction $\bigwedge_{i < m} \varphi(x, b; d_i)$ would be inconsistent. But this contradicts the fact that $\theta(b; \bar{d})$ holds. Hence we may find $a \in \mathfrak{U}$ such that $\varphi(a, b; \bar{d})$ holds and $\text{tp}_{\tilde{\mathcal{L}}}(a, b/N)$ does not fork over M (equivalently is $\tilde{\mathcal{L}}(M)$ -invariant). \blacksquare

Lemma 2.4:

Let $p(x)$ be a global \mathcal{L} -type which is $\tilde{\mathcal{L}}(M)$ -invariant. Then p is $\tilde{\mathcal{L}}(M)$ -definable if and only if for every M -finitely satisfiable $\tilde{\mathcal{L}}$ -type $q(y)$, $p(x) \otimes q(y)|_{\mathcal{L}(M)} = q(y) \otimes p(x)|_{\mathcal{L}(M)}$.

Proof. Assume that p is $\tilde{\mathcal{L}}(M)$ -definable. Let $q(y)$ be a global $\tilde{\mathcal{L}}$ -type, finitely satisfiable in M and let $\varphi(x; y) \in \mathcal{L}(M)$. Let $d\varphi(y) \in \tilde{\mathcal{L}}(M)$ be the φ -definition of p . Let $a \models p|_M$. Then for every $b \in M$, we have $d\varphi(b) \Leftrightarrow \varphi(a; b)$. Therefore by finite satisfiability of q , we have $d\varphi(y) \in q \Leftrightarrow \varphi(a; y) \in q$. On the other hand, we have $p(x) \otimes q(y) \vdash \varphi(x; y)$ if and only if $q(y) \vdash d\varphi(y)$. Hence we see that $p(x) \otimes q(y) \vdash \varphi(x; y) \Leftrightarrow q \vdash d\varphi(y) \Leftrightarrow q(y) \otimes p(y) \vdash \varphi(x; y)$.

Conversely, assume that p commutes with every M -finitely satisfiable $\tilde{\mathcal{L}}$ -type as in the statement of the lemma. Let $\varphi(x; y) \in \mathcal{L}(M)$. Fix a type $q_0 \in S_y^{\tilde{\mathcal{L}}}(M)$. Assume for example that $p(x) \otimes q_0(y) \vdash \varphi(x; y)$. Then for every global coheir $q(y)$ of $q_0(y)$, we have $q(y) \otimes p(x) \vdash \varphi(x; y)$. This easily implies that there is some formula $d\varphi_{q_0}(y) \in q_0(y)$ such that $p \vdash \varphi(x; b)$ for every $b \in d\varphi_{q_0}(M)$. But then for every $q_1(y) \in S_y^{\tilde{\mathcal{L}}}(M)$, if $q_1(y) \vdash d\varphi_{q_0}(y)$, then $p(x) \otimes q_1(y) \vdash \varphi(x; y)$ (applying commutativity again). Hence the set of types $q \in S_y^{\tilde{\mathcal{L}}}(M)$ for which $p(x) \otimes q(y) \vdash \varphi(x; y)$ is open. Applying this to $\neg\varphi$ instead of φ shows that it is also closed. Therefore p is $\tilde{\mathcal{L}}(M)$ -definable. \blacksquare

The following lemma appears as [9, Lemma 2.6].

Lemma 2.5:

<We work in \mathcal{L} > Let $B \subset \mathfrak{U}$ and let $a \in \mathfrak{U}$ be a tuple such that $\text{dp-rk}(a/B) = 1$. Let \bar{b} in \mathfrak{U} be an infinite sequence, indiscernible over B but not over Ba . Let $\varphi(x; y) \in \mathcal{L}$, $|x| = |a|$. Then there are formulas $\psi(x) \in \text{tp}(a/B\bar{b})$ and $\theta_l(y) \in \mathcal{L}(B\bar{b})$ $l = 0, 1$, such that:

- ₀ for each $b \in B^{|y|}$, one of $\theta_0(b)$ or $\theta_1(b)$ holds;

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- ₁ for $l = 0, 1$, $\mathfrak{U} \models \theta_l(y) \wedge \psi(x) \rightarrow \varphi^l(x; y)$.

As in [9], we deduce the following.

Lemma 2.6:

Let $M <^+ N_1 <^+ N$ in the sense of \tilde{T} and let $a \in \mathfrak{U}$ such that $\text{dp-rk}_{\mathcal{L}}(a/N) = 1$. Let \bar{b} in \mathfrak{U} be an infinite sequence \mathcal{L} -indiscernible over N , which is not \mathcal{L} -indiscernible over Na . Assume also that $\text{tp}_{\mathcal{L}}(a\bar{b}/N)$ is $\tilde{\mathcal{L}}(M)$ -invariant. Then

$$\text{tp}_{\mathcal{L}}(a/N_1\bar{b}) \vdash \text{tp}_{\mathcal{L}}(a/N).$$

More precisely, given $\varphi(x; y) \in \mathcal{L}$, $|x| = |a|$, there are formulas $\theta_l(y) \in \mathcal{L}(N_1\bar{b})$ ($l = 0, 1$) and $\psi(x) \in \text{tp}_{\mathcal{L}}(a/N_1\bar{b})$ such that:

- ₀ for each $b \in N^{|y|}$, one of $\theta_0(b)$ or $\theta_1(b)$ holds;
- ₁ for $l = 0, 1$, $\mathfrak{U} \models \theta_l(y) \wedge \psi(x) \rightarrow \varphi^l(x; y)$.

Proof. Let $\psi(x), \theta_l(y) \in \mathcal{L}(N)$ be given by Lemma (2.5) with $B = N$.

Write $\theta_l(y) = \theta_l(y; \bar{b}, e)$ and $\psi(x) = \psi(x; \bar{b}, e)$ with $e \in N$. As $\text{tp}_{\mathcal{L}}(a\bar{b}/N)$ is $\tilde{\mathcal{L}}(M)$ -invariant, we may replace e by any $e' \in N$ such that $\text{tp}_{\tilde{\mathcal{L}}}(e'/M) = \text{tp}_{\tilde{\mathcal{L}}}(e/M)$. In particular, we may assume that $e \in N_1$. This gives what we want. ■

Proposition 2.7:

Let p be a global \mathcal{L} -type of $\text{dp-rank } 1$. Assume that p is $\tilde{\mathcal{L}}(M)$ -invariant. Then p is either finitely satisfiable in M or $\tilde{\mathcal{L}}(M)$ -definable.

Proof. Assume that p is not $\tilde{\mathcal{L}}(M)$ -definable. Then there is a global $\tilde{\mathcal{L}}$ -type q finitely satisfiable in M such that p does not commute with q as in Lemma (2.4). Take $N > M$ sufficiently saturated. Let $\varphi(x; y) \in \mathcal{L}$, $d \in N$ such that $\varphi(x; d) \in p$.

Let $(a, b) \models p \otimes q|_{\mathcal{L}(N)}$, then let I be a Morley sequence of q over Nab and let $\bar{b} = b + I$. In the reduct to \mathcal{L} , the sequence \bar{b} is indiscernible over N , but not over Na . Let $M <^+ N_1 <^+ N$ with $\text{tp}_{\tilde{\mathcal{L}}}(N_1/Md)$ finitely satisfiable in M .

Apply Lemma (2.6) to a, N_1, N and \bar{b} . The second part of the conclusion gives formulas $\theta_l(y), \psi(y) \in \mathcal{L}(N_1\bar{b})$. Write $\theta_l(y) = \theta_l(y; \bar{b}, e)$ and $\psi(x) = \psi(x; \bar{b}, e)$ with $e \in N_1$.

Since $\varphi(x; d) \in p$, we know that the formula $\theta_1(d; \bar{b}, e)$ holds. As $\text{tp}_{\mathcal{L}}(\bar{b}/N)$ is finitely satisfiable in M , there is $\bar{b}_0 \in M$ such that

$$\bar{b}_0 \models \theta_1(d; \bar{z}, e) \wedge (\exists x)(\forall y)(\theta_1(y; \bar{z}, e) \Rightarrow \varphi(x; y)).$$

Since N_1 is a model, there is $a_0 \in N_1$ such that $(\forall y)(\theta_1(y; \bar{b}_0, e) \rightarrow \varphi(a_0; y))$ holds. In particular $\varphi(a_0; d)$ holds. As $\text{tp}_{\mathcal{L}}(N_1/Md)$ is finitely satisfiable in M , we can find $a'_0 \in M$ satisfying $\varphi(x; d)$. As $\varphi(x; d)$ was any formula in p , this proves that p is finitely satisfiable in M . ■

Proposition 2.8:

Assume that \tilde{T} is NIP and that T is dp-minimal and has property (D). Let $M \models \tilde{T}$ and $\varphi(x; d) \in \tilde{\mathcal{L}}(\mathfrak{U})$ be non-forking over M . Then there is a complete $\tilde{\mathcal{L}}(M)$ -definable \mathcal{L} -type, consistent with $\varphi(x; d)$.

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Proof.

The proof is exactly as in [10], replacing \mathcal{L} by $\tilde{\mathcal{L}}$ when necessary. We argue by induction on the length of the variable x .

$|x| = 1$: Assume that $|x| = 1$ and take $p(x)$ a global \mathcal{L} -type consistent with $\varphi(x; d)$ and non-forking over M in the sense of \tilde{T} . If p is $\tilde{\mathcal{L}}(M)$ -definable, we are done. Otherwise, by Proposition (2.7), p is finitely satisfiable in M . This implies that $\varphi(x; d)$ has a solution a in M . Then we can take $\text{tp}_{\mathcal{L}}(a/\mathfrak{U})$.

Induction: Assume we know the result for $|x| = n$, and consider a non-forking formula $\varphi(x_1, x_2; d) \in \tilde{\mathcal{L}}(\mathfrak{U})$, where $|x_2| = n$ and $|x_1| = 1$. Let $N \succ M$ sufficiently saturated, with $d \in N$. Using the induction hypothesis and Lemma (2.3), we can find a tuple $(a_1, a_2) \models \varphi(x_1, x_2; d)$ such that $\tilde{p} := \text{tp}_{\tilde{\mathcal{L}}}(a_1, a_2/N)$ is $\tilde{\mathcal{L}}(M)$ -invariant and $\text{tp}_{\mathcal{L}}(a_2/N)$ is $\tilde{\mathcal{L}}(M)$ -definable.

If $p = \text{tp}_{\mathcal{L}}(a_1, a_2/N)$ is $\tilde{\mathcal{L}}(M)$ -definable we are done. Otherwise, there is some type $q \in S_y^{\tilde{\mathcal{L}}}(N)$ finitely satisfiable in M such that p does not commute with q as in Lemma (2.4).

Now let $c \in \mathfrak{U}$ such that $(a_1 \hat{~} a_2, c) \models \tilde{p} \otimes q$. Let I be a Morley sequence of q over everything. As $\text{tp}_{\mathcal{L}}(a_2/N)$ is definable, it commutes with q . Therefore, the sequence $\bar{c} = c + I$ is \mathcal{L} -indiscernible over Na_2 . However, it is not \mathcal{L} -indiscernible over Na_1a_2 . Take some $M \prec^+ N_1 \prec^+ N$ with $\text{tp}_{\tilde{\mathcal{L}}}(N_1/Md)$ finitely satisfiable in M .

From now on, unless we say explicitly otherwise, we work in \mathcal{L} . Take $r \in S(\mathfrak{U})$ finitely satisfiable in N . Let $b \models r|_{Na_2\bar{c}}$. Build a Morley sequence J of r over everything. Then $b + J$ is indiscernible over $Na_2\bar{c}$ and \bar{c} is indiscernible over Na_2bJ . As \bar{c} is not indiscernible over Na_1a_2 , by dp-minimality, $b + J$ must be indiscernible over Na_1a_2 . Hence $b \models r|_{Na_1a_2\bar{c}}$. We have shown that $r|_{Na_2\bar{c}} \vdash r|_{Na_1a_2\bar{c}}$. Let $l = l_r \in \{0, 1\}$ such that $r(y) \vdash \varphi^l(a_1, a_2; y)$. Then $r(y)|_{Na_2\bar{c}} \vdash \varphi^l(a_1, a_2; y)$. By compactness, there is a formula $\theta_r(y)$ in $r(y)|_{Na_2\bar{c}}$ which already implies $\varphi^l(a_1, a_2; y)$. Using compactness of the space of global N -finitely satisfiable types, we can extract from the family $(\theta_r(y))_r$ a finite subcover \mathcal{C} . Let $\theta_l(y)$ be the disjunction of the formulas in \mathcal{C} that imply $\varphi^l(a_1, a_2; y)$. Summing up, we have: $\mathfrak{U} \models \theta_l(y) \Rightarrow \varphi^l(a_1, a_2; y)$, $l = 0, 1$, and every type finitely satisfiable in N satisfies either $\theta_1(y)$ or $\theta_2(y)$. In particular, this is true of any point $n \in N$.

Write $\theta_1(y)$ as $\theta_1(y; a_2, \bar{c}, e)$ exhibiting all parameters, with $e \in N$. By invariance of $\text{tp}_{\tilde{\mathcal{L}}}(a_1, a_2, \bar{c}/N)$, we may assume that $e \in N_1$ and in particular $\text{tp}_{\tilde{\mathcal{L}}}(e/Md)$ is finitely satisfiable in M .

As $\text{tp}(\bar{c}/Na_2)$ is finitely satisfiable in M , there is $\bar{c}' \in M$ such that:

$$\models \theta_1(d; a_2, \bar{c}', e) \wedge (\exists x_1)(\forall y)(\theta_1(y; a_2, \bar{c}', e) \Rightarrow \varphi(x_1, a_2; y)).$$

Now, $\text{tp}_{\tilde{\mathcal{L}}}(e/Md)$ is finitely satisfiable in M . As $\text{tp}_{\mathcal{L}}(a_2/N)$ is $\tilde{\mathcal{L}}(M)$ -definable, also $\text{tp}_{\mathcal{L}}(e/Mda_2)$ is finitely satisfiable in M and we may find $e' \in M$ such that the previous formula holds with e replaced by e' .

By property (D), there is some $\mathcal{L}(Ma_2)$ -definable \mathcal{L} -type $p_1(x_1) \in S(Ma_2)$ containing the formula $(\forall y)(\theta_1(y; a_2, \bar{c}', e') \Rightarrow \varphi(x_1, a_2; y))$. By Lemma (2.2), p_1 extends to a complete $\tilde{\mathcal{L}}(Ma_2)$ -definable \mathcal{L} -type over Na_2 . Let a'_1 realise that type. Then $\text{tp}_{\mathcal{L}}(a'_1, a_2/N)$ is $\tilde{\mathcal{L}}(M)$ -definable and we have $\models \varphi(a'_1, a_2; d)$ as required. \blacksquare

3 Valued differential fields

The main motivation for the results in the previous sections was to understand definable and invariant types in valued differential fields and more specifically those where the derivation preserves the valuation, i.e. for all x , $\text{val}(\partial(x)) \geq \text{val}(x)$. In [6], Scanlon showed that the theory of valued fields with a valuation preserving derivation has a model completion named $\text{VDF}_{\mathcal{EC}}$. It is the theory of so called ∂ -Henselian fields whose residue fields is a model of DCF_0 , whose valued group is divisible and such that for all x there exists a y with $\partial(y) = 0$ and $\text{val}(y) = \text{val}(x)$. One can refer to [6] for a precise description of this theory.

The main result that we will be needing here is that the theory $\text{VDF}_{\mathcal{EC}}$ eliminates quantifiers in the one sorted language $\mathcal{L}_{\partial, \text{div}}$ consisting of the language of rings enriched with a symbol ∂ for the derivation and a symbol $x|y$ interpreted as $\text{val}(x) \leq \text{val}(y)$. This result implies that for all substructures $A \leq M \models \text{VDF}_{\mathcal{EC}}$ the map sending $p = \text{tp}_{\mathcal{L}_{\partial, \text{div}}}(c/A)$ to $\nabla_\omega p := \text{tp}_{\mathcal{L}_{\text{div}}}((\partial^i(c))_{i \in \omega}/A)$ is injective, where $\mathcal{L}_{\text{div}} := \mathcal{L}_{\partial, \text{div}} \setminus \{\partial\}$ denotes the one sorted language of valued fields.

Lemma 3.1:

Let $k \models \text{DCF}_0$. The Hahn field $k((t^{\mathbb{R}}))$ is a models of $\text{VDF}_{\mathcal{EC}}$ and its reduct to \mathcal{L}_{div} is uniformly stably embedded in every elementary extension.

Proof. The fact that $k((t^{\mathbb{R}})) \models \text{VDF}_{\mathcal{EC}}$ follows from the fact that its residue field k is a model of DCF_0 , its value group \mathbb{R} is a divisible ordered abelian group and that Hahn fields are spherically complete, cf. [6, Proposition 6.1].

The fact that $k((t^{\mathbb{R}}))$ is uniformly stably embedded in every elementary extension is shown in [5, Corollary A.7]. ■

Recall that Haskell, Hrushovski and Macpherson [3] showed that algebraically closed valued fields eliminate imaginaries provided the so-called geometric sorts are added. We will be denoting by \mathcal{G} the set of all geometric sorts.

Proposition 3.2:

Let $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M \models \text{VDF}_{\mathcal{EC}}$. A type $p \in \mathcal{S}^{\mathcal{L}_{\text{div}}}(M)$ is $\mathcal{L}_{\partial, \text{div}}(A)$ -definable if and only if it is $\mathcal{L}_{\text{div}}(\mathcal{G}(A))$ -definable.

Proof. If p is $\mathcal{L}_{\text{div}}(\mathcal{G}(A))$ -definable then it is in particular $\mathcal{L}_{\partial, \text{div}}(A)$ -definable. The reciprocal implication follows immediately from Corollary (1.7) and Lemma (3.1). ■

An immediate corollary of this proposition is an elimination of imaginaries result for canonical bases of definable types in $\text{VDF}_{\mathcal{EC}}$:

Corollary 3.3:

Let $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M \models \text{VDF}_{\mathcal{EC}}$ and $p \in \mathcal{S}^{\mathcal{L}_{\partial, \text{div}}}(M)$. The following are equivalent:

- (i) p is $\mathcal{L}_{\partial, \text{div}}(A)$ -definable;
- (ii) $\nabla_\omega(p)$ is $\mathcal{L}_{\text{div}}(\mathcal{G}(A))$ -definable;

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(iii) p is $\mathcal{L}_{\partial, \text{div}}(\mathcal{G}(A))$ -definable.

Proof. The implication (iii) \Rightarrow (i) is trivial. Let us now assume (i). An $\mathcal{L}_{\text{div}}(M)$ -formula $\varphi(\bar{x}; m)$ is in $\nabla_\omega(p)$ if and only if $\varphi(\partial_\omega(x); m) \in p$. It follows that $\nabla_\omega(p)$ is $\mathcal{L}_{\partial, \text{div}}(A)$ -definable. By Proposition (3.2), $\nabla_\omega(p)$ is in fact $\mathcal{L}_{\text{div}}(\mathcal{G}(A))$ -definable. Let us now assume (ii) and let $\psi(x; m)$ be any $\mathcal{L}_{\partial, \text{div}}(M)$ -formula. By quantifier elimination, $\psi(x; m)$ is equivalent to $\varphi(\partial_\omega(x); \partial_\omega(m))$ for some \mathcal{L}_{div} -formula $\varphi(\bar{x}; \bar{t})$. Therefore $\psi(x; m) \in p$ if and only if $\varphi(\bar{x}; \partial_\omega(m)) \in \nabla_\omega(p)$ and hence p is $\mathcal{L}_{\partial, \text{div}}(\mathcal{G}(A))$ -definable. ■

In [5], we will show that there are enough definable types to use this partial elimination of imaginaries result to obtain elimination of imaginaries to the geometric sorts for $\text{VDF}_{\mathcal{EC}}$. Thanks to the result in Section 1 and results from [5], we can also characterise invariant types in $\text{VDF}_{\mathcal{EC}}$. Note that, although the main results in [5] depend on the results proved in the present paper, the result from [5] that we will be using in what follows does not.

Proposition 3.4:

Let $M \models \text{VDF}_{\mathcal{EC}}$ and $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M^{\text{eq}}$. A type $p \in \mathcal{S}^{\mathcal{L}_{\text{div}}}(M)$ is $\mathcal{L}_{\partial, \text{div}}(A)$ -invariant if and only if it is $\mathcal{L}_{\text{div}}(\mathcal{G}(A))$ -invariant.

Proof. To prove the non obvious implication, by Proposition (1.9), we have to show that $\text{VDF}_{\mathcal{EC}}$ has a model whose underlying valued field is uniformly stably embedded in any elementary extension — that is tackled in Lemma (3.1) — and that any $\mathcal{L}_{\partial, \text{div}}(A)$ -definable set is consistent with an $\mathcal{L}(\mathcal{G}(A))$ -invariant \mathcal{L} -type. It follows from [5, Corollary 9.7] (applied to $T = \text{ACVF}$ and $\tilde{T} = \text{VDF}_{\mathcal{EC}}$) that any $\mathcal{L}_{\partial, \text{div}}(A)$ -definable set is consistent with an $\mathcal{L}_{\partial, \text{div}}(A)$ -definable \mathcal{L}_{div} -type. But, by Proposition (3.2), such a type is $\mathcal{L}_{\text{div}}(\mathcal{G}(A))$ -definable. ■

Corollary 3.5:

Let $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M \models \text{VDF}_{\mathcal{EC}}$ and $p \in \mathcal{S}^{\mathcal{L}_{\partial, \text{div}}}(M)$. The following are equivalent:

- (i) p is $\mathcal{L}_{\partial, \text{div}}(A)$ -invariant;
- (ii) $\nabla_\omega(p)$ is $\mathcal{L}_{\text{div}}(\mathcal{G}(A))$ -invariant;
- (iii) p is $\mathcal{L}_{\partial, \text{div}}(\mathcal{G}(A))$ -invariant.

Proof. This is proved as in Corollary (3.3), except that Proposition (3.4) is used instead of Proposition (3.2). ■

We can now give a characterisation of forking in $\text{VDF}_{\mathcal{EC}}$.

Corollary 3.6:

Let $M \models \text{VDF}_{\mathcal{EC}}$ be $|A|^+$ -saturated, $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M$ and $\varphi(x)$ be an $\mathcal{L}_{\partial, \text{div}}(M)$ -formula. Then $\varphi(x)$ does not fork over A if and only if for all $\mathcal{L}_{\text{div}}(M)$ -formulas such that $\varphi(x)$ is equivalent to $\psi(\partial_\omega(x))$, $\psi(\bar{x})$ does not fork over $\mathcal{G}(A)$ (in ACVF).

Proof. Let us first assume that $\varphi(x)$ does not fork over A and let p be a global non forking extension of $\varphi(x)$. As $\text{VDF}_{\mathcal{EC}}$ is NIP, by [4, Proposition 2.1], p is invariant under

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all automorphisms that fix Lascar strong type over A . But, because $\text{VDF}_{\mathcal{EC}}$ has the invariant extension property (cf. [5, Theorem A]), Lascar strong type and strong type coincide in $\text{VDF}_{\mathcal{EC}}$, hence p is $\mathcal{L}_{\partial, \text{div}}(A)$ -invariant. It follows from Corollary (3.5) that $\nabla_{\omega}(p)$ is $\mathcal{L}_{\text{div}}(\mathcal{G}(A))$ -invariant and hence $\psi(\bar{x})$ does not fork over $\mathcal{G}(A)$. Let us now assume that no $\psi(\bar{x})$ such that $\varphi(x)$ is equivalent to $\psi(\partial_{\omega}(x))$ forks over $\mathcal{G}(A)$. Then there exists $q \in \mathcal{S}_{\bar{x}}^{\mathcal{L}_{\text{div}}}(M)$ which is $\mathcal{L}_{\text{div}}(\mathcal{G}(A))$ -invariant and consistent with all such formulas $\psi(\bar{x})$. Now, the image of the continuous map $\nabla_{\omega} : \mathcal{S}_x^{\mathcal{L}_{\partial, \text{div}}}(M) \rightarrow \mathcal{S}_{\bar{x}}^{\mathcal{L}_{\text{div}}}(M)$ is closed and if $\chi(\bar{x})$ is an $\mathcal{L}_{\text{div}}(M)$ -formula containing the image of ∇_{ω} and $\psi(\bar{x})$ is as above, $\chi(\partial_{\omega}(x)) \wedge \psi(\partial_{\omega}(x))$ is also equivalent to $\varphi(x)$. Therefore, $q = \nabla_{\omega}(p)$ for some $\mathcal{L}_{\partial, \text{div}}(A)$ -invariant $p \in \mathcal{S}_x^{\mathcal{L}_{\partial, \text{div}}}(M)$. This type p implies $\varphi(x)$ and hence $\varphi(x)$ does not fork over A . ■

Remark 3.7:

The previous corollary is somewhat unsatisfying as one needs to consider all possible way of describing $\varphi(x)$ as the prolongation points of an \mathcal{L}_{div} -formula ψ (with parameters in a saturated model) to conclude whether φ forks or not.

But clearly considering only one such ψ cannot be enough. For example, consider any definable set $\varphi(x)$ forking (in $\text{VDF}_{\mathcal{EC}}$) over A and let $\psi(x_0, x_1) = (\text{val}(x_0) \geq 0 \wedge \text{val}(x_1) < 0) \vee \varphi(x_0)$. Then the set $\{x \in M : M \models \psi(x, \partial(x))\} = \varphi(M)$ but ψ does not fork over A (in ACVF).

The obstruction here might seem frivolous, but it is the core of the problem. Indeed, it is not clear if there is a way, given φ to find a formula ψ as above that does not contain “large” subsets with no prolongation points.

In fact, using the results Section 2, over a model we can say a bit more about non forking formulas in $\text{VDF}_{\mathcal{EC}}$:

Proposition 3.8:

Let $M \preceq N \models \text{VDF}_{\mathcal{EC}}$ and $\varphi(x)$ be an $\mathcal{L}_{\partial, \text{div}}(N)$ -formula that does not fork over M , then there exists an $\mathcal{L}_{\text{div}}(M)$ -definable \mathcal{L}_{div} -type consistent with φ .

Proof. This is an immediate consequence of Propositions (2.8) and (3.2). ■

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